CLASS: Msc MATHEMATICS II SEM

SUBJECT CODE: H-2051

SUBJECT NAME: ADVANCED DISCRETE MATHEMATICS

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01 / ADM — CH-07 — Dr. Satish Kr Gupta

Tree

A connected graph without any circuit is called a tree.

example.

Tree.

Note. Since parallel edge and self loop form a circuit so a tree can not have a parallel edge and self loop.

Theorem. (1) A graph G is a tree iff there is one and only one path between any two vertices of G.

Proof: "If part" suppose G is a tree

To prove there is one and only path between any two vertices of G.

It is given G is a tree $\Rightarrow$ G is connected graph without any circuit.

$\Rightarrow$ there must exist at least one path between any two vertices of G.

Suppose there exists two distinct paths between vertices a and b of G. Then the union of those two paths will form a circuit and then G is not a tree

Hence there is one and only one path between any two vertices of G

Only if part: suppose that there is one and only one path between any two vertices of G.

To show G is a Tree

It is given there is one and only one path between any two vertices of G

$\Rightarrow$ G is a connected graph without any circuit.

$\Rightarrow$ G is a tree

Theorem. 02 A tree with n vertices has $(n-1)$ edges.

Proof. We shall prove the theorem by induction on the number of vertices. Clearly, the theorem is true for trees with one or two vertices

as

Assum that theorem is true for all trees with less than n vertices. —— (1)

Let us consider a tree G with n vertices. Let e_k be any edge in G with end vertices v_i and v_j.

$\Rightarrow$ the edge e_k is the only path between v_i and v_j.

$\Rightarrow$ $G - e_k$ is disconnected

$\Rightarrow$ $G - e_k$ will contain exactly two components

let G_1 and G_2 be two components of $G - e_k$

$\Rightarrow$ G_1 is a tree & G_2 is a tree as G has no circuit

Let n_1 be no of vertices in G_1

n_2 be no of vertices in G_2

by hypothesis

no of edges in G_1 $= n_1 - 1$

no of edges in G_2 $= n_2 - 1$

$\therefore$ no of edges in $G - e_k = n_1 - 1 + n_2 - 1 = n - 2$ edges $[n = n_1 + n_2]$

Hence G has exactly $(n-1)$ edges [including e_k]

proved.

Theorem 3. Every connected graph with n vertices and $(n-1)$ edges is a tree.

Proof. Let G be a connected graph with n vertices and $(n-1)$ edges

To prove G is a tree i.e G has no circuit

Suppose if possible G has atleast one circuit

Since removing an edge from a circuit, we get connected graph

if G has two circuits we remove two edges & we get again connected graph

In this way we continue to remove the edges until G is circuit free

let G^* be the resulting graph. ie connected graph

$\Rightarrow G^*$ is a tree

$\Rightarrow G^*$ has $(n-1)$ edges + many more edges that we remove to make G circuitless

$\Rightarrow G^*$ has more than $(n-1)$ edges

which is a contradiction

Hence G has no circuit

$\Rightarrow G$ is a tree

Theorem 4. A graph G with n vertices, $(n-1)$ edges and no circuits is a tree.

Proof. Let G be a graph with n vertices and $(n-1)$ edges and has no circuit.

$\Rightarrow G$ will be a tree if G is connected

Suppose if possible G is disconnected.

$\Rightarrow G$ has two or more than two circuitless components

Let G_1 and G_2 be any two such components of G

We add an edge between a vertex v_1 in G_1 and v_2 in G_2

Since u_1 and u_2 are in different components of G
$\Rightarrow$ there is no path between u_1 and u_2 because G is disconnected
and addition of e will not create a circuit
$\Rightarrow$ $G \cup \{e\}$ is a circuitless
$\Rightarrow$ $G \cup \{e\}$ is connected & without circuit
$\Rightarrow$ $G \cup \{e\}$ has n vertices & n edges
which is a contradiction
$\Rightarrow$ G must be connected
$\Rightarrow$ G is a tree

Minimally connected graph

A connected graph G is said to be minimally connected if removal of any edge from G disconnects the graph G.

Theorem. 5. A graph G is a tree iff it is minimally connected.

Proof. "If part" Suppose that G is a tree
To show G is minimally connected
Suppose if possible G is not minimally connected
$\Rightarrow$ $\exists$ $e \in G$ such that $G - \{e\}$ is connected
$\Rightarrow$ e is in some circuit
$\Rightarrow$ G has a circuit
$\Rightarrow$ But G is a tree $\Rightarrow$ we have a contradiction
So, G must be minimally connected.

"Only if part" Suppose G is minimally connected
To prove G is a tree

It is given G is minimally connected
$\Rightarrow$ G is connected & has no circuit
$\Rightarrow$ G is a tree

Theorem. 06. In any tree with two or more vertices there are atleast two pendant vertices.

Proof. Let G be any tree having n vertices.
Then G has (n-1) edges
Since each edge contributes two to the sum of degree of the vertices
i.e $\sum_{i=1}^{n} d(v_i) = 2(n-1)$ [$\because$ e = n-1, here]
Here 2(n-1) are to be divided among n vertices in G
Let the number of vertices of degree one in G be x , we have

$$\frac{2(n-1)-x}{n-x} \geq 2$$

$$\cancel{2n}-2-x \geq \cancel{2n}-2x$$

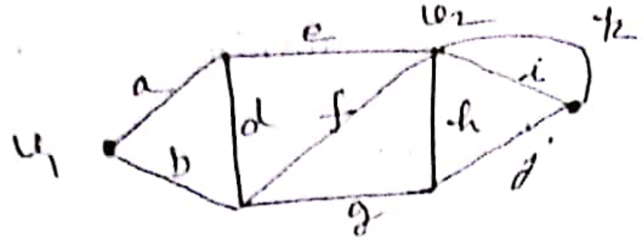
$$x \geq 2$$

$\Rightarrow$ no of pendant vertices are atleast two in a tree.

Distance and Centres in a tree

Let G be a connected graph and v_i, v_j be any two vertices in G. The distance $d(v_i, v_j)$ between the vertices v_i and v_j is the length of the shortest path. i.e no of edges in the shortest path

Example.



Paths between u_1 & u_2 are

(a, e), (a, d, f), (b, f), (b, g, h), ...

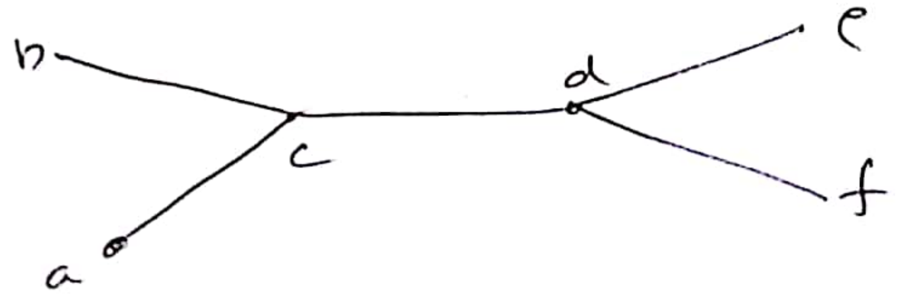
Length of shortest paths

$d(u_1, u_2) = (a, e)$
$= 2$ [no of edges]

Eccentricity of a vertex The eccentricity $E(u)$ of a vertex in a graph G is the distance between u and the vertex u_i farthest from u in G

symbolically $E(u) = \max_{u_i \in G} d(u, u_i)$

ex



$E(a) = \max\{d(a,b), d(a,c), d(a,d), d(a,e), d(a,f)\}$

$= \max\{2, 1, 2, 3, 3\}$

$E(a) = 3$

$E(b) = \max\{d(b,c), d(b,a), d(b,d), d(b,e), d(b,f)\}$

$E(b) = \max\{1, 2, 2, 3, 3\} = 3$

$E(c) = \max\{d(c,a), d(c,b), d(c,d), d(c,e), d(c,f)\}$
$= \max\{1, 1, 1, 2, 2\} = 2$

$E(d) = \max\{d(d,a), d(d,b), d(d,c), d(d,e), d(d,f)\}$

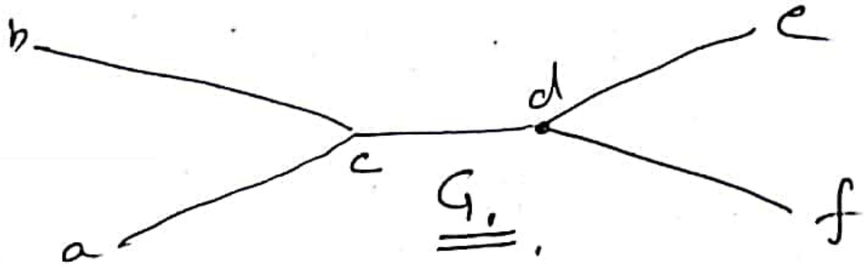
$= \max\{2, 2, 1, 1, 1\} = 2$

[किसी भी vertex की Eccentricity से मतलब है कि उस vertex से हम दिये हुए graph में edge द्वारा कितनी दूर जा सकते हैं अर्थात दूर से दूर तक जाने में edge की संख्या कितनी है ~~मक~~ यही संख्या Eccentricity कहलाती है]

centre of a graph By centre of a graph we mean a vertex whose Eccentricity is minimum

Note. centre of a graph need not be unique these may be more than one

example.



G.

Here $E(c) = 2$ $\quad E(a) = 3$ $\quad E(e) = 3$

$E(d) = 2$ $\quad E(b) = 3$ $\quad E(f) = 3$

Here 2 is minimum

$\Rightarrow$ c and d are The centre of The given graph G.

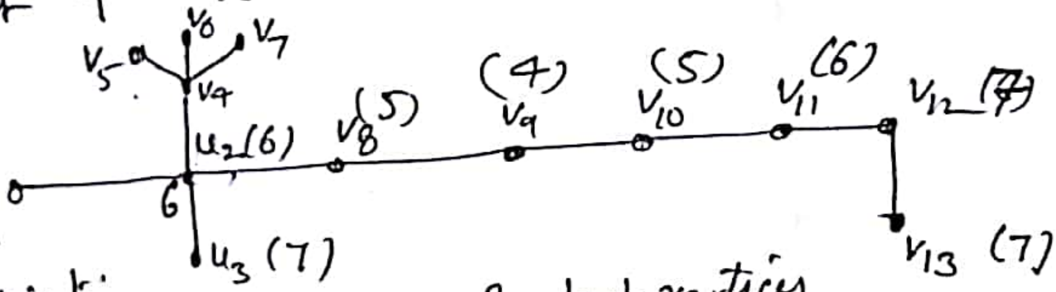
Theorem. (1) Every tree has either one or two centres

Proof. We know that it is obvious the maximum distance, max. $d(u, v_i)$ from a given vertex u to any other vertex v_i occurs only when v_i is a Pendant vertex (i.e a vertex of degree one).

Let T be a tree having more than two vertices
$\Rightarrow$ T must have atleast two pendant vertices
Remove all the Pendant vertices from T to obtain new graph T'. Then T' is a tree
$\Rightarrow$ T' must have atleast two pendant vertices
Remove all the Pendant vertices from T' to obtain T''. Continue this process of removal of pendant vertices. Ultimately we reach to either a vertex (i.e a graph having only one vertex) which is a centre or an edge i.e a graph whose end vertices are the centre.
Hence Every tree has either one or two centres.

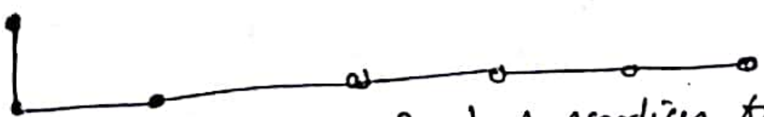
Practical example

Ex Let T is a tree as



Eccentricity (E)

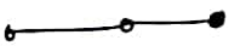
Remove all Pendant vertices from T

T' : 

Remove all Pendant vertices from T'

T'' : 

similarly

T''' : 

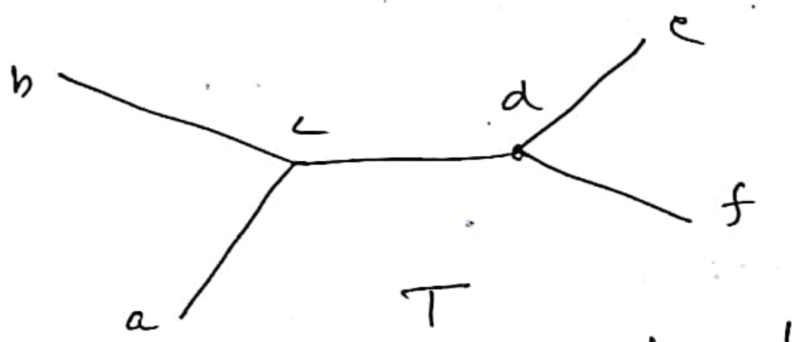
Then T^{IV} : 

$\Rightarrow$ T^{IV} has two centres, Proved

Radius of a tree. Radius of a tree is the distance from the centre to the farthest vertex.

Diamete of a tree. The diameter of a tree T is defined as the length of the longest path in T

Example



Radius of T = distance from c or from d to the fathest vertex of T

= 2 (i.e Eccentricity of the ~~centre~~ centre of T)

Diameter of T = length of longest path in T

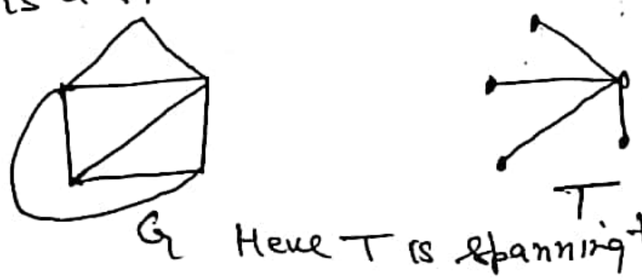
= length of b, c, d, f

= 3

Note. Here it is clear diameter of a tree is not the double of its radius.

Spanning tree A subgraph T of a connected graph G is said to be spanning tree of G if T is a tree and contains all vertices of G.

Example.



Here T is spanning tree of G.

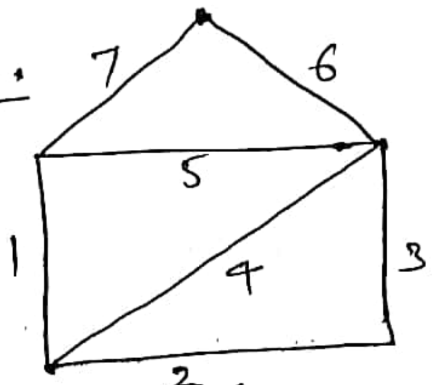
forest. A collection of spanning tree of G is called a forest.

Note. There may be more than one spanning tree of G.

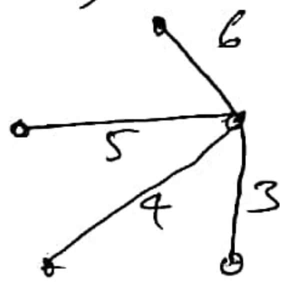
Branches of a spanning tree T. Branches of a spanning tree we mean the number of edges in T of G.

Chords of T: By chords of T we mean the number of edges that are not in T but in G.
(Here T we mean spanning tree of G).

Example.



G



Spanning Tree T of G

Branches of T are 3, 4, 5, 6

Chordes of T are 1, 2, 7

Note. Let T be a spanning tree of a graph G having n vertices and $(n-1)$ edges such that G contains e edges then

- no of Branches of T = $(n-1)$

no of chords for T = $e-(n-1) = (e-n+1)$

Rank and Nulity

Consider a graph G with n vertices, e edges and k components.

The rank of graph G is defined as

rank $\boxed{r = n-k}$

and the nulity of the graph G

nulity $\mu = e-n+k$

$\boxed{\mu = e-r}$

Note ① rank + nulity = no of edges in Graph G

② If G is connected, $k=1$
then rank of G = $(n-1)$
i.e $\boxed{r=(n-1)}$ = no of branches in spanning tree T of G

$\mu = e-(n-1)$

$\boxed{\mu = (e-n+1)}$

= no of chords in G.

③ Fundamental circuit. A circuit formed by adding a chord to a spanning tree T of G (connected graph) is called a fundamental circuit

so no of fundamental circuits in a connected graph G = $(e-n+1)$, e is no of edges in G and n is no of vertices in G

Minimal Spanning Trees

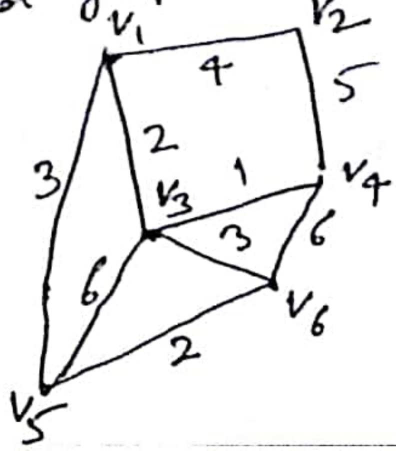
Let G be a weighted graph (weighted graph we mean in which a non-negative number is assigned to the each edge. The weight of the spanning tree T of G is defined as the sum of the weights of all branches in T.

In general, different spanning trees of a weighted graph will have different weight.

A spanning tree T of weighted graph G is called minimal spanning tree if the weight of T is smallest.

Note. There are two methods namely Kruskal's method and Prim's method to find the minimal spanning T of a weighted graph but here we explain only Kruskal method.

Question. Find the minimal spanning tree T using ~~KS~~ Kruskal's method of the following weighted graph.



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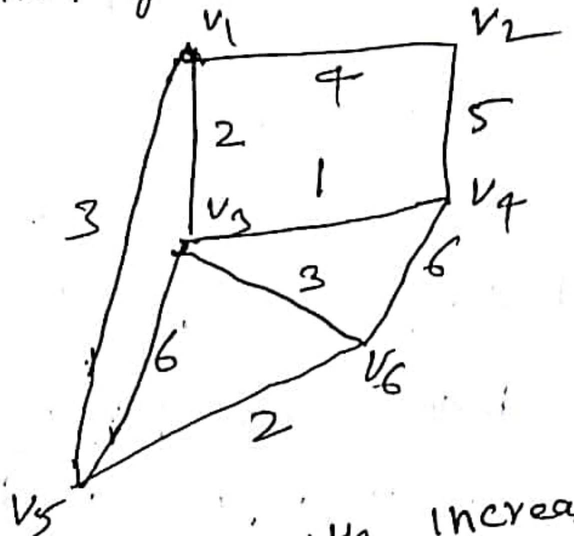
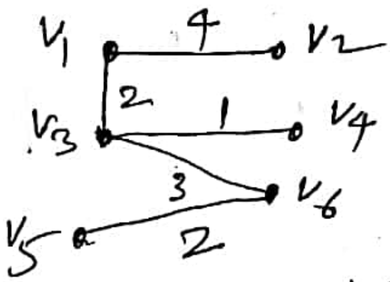
Sol^n Given weighted graph is



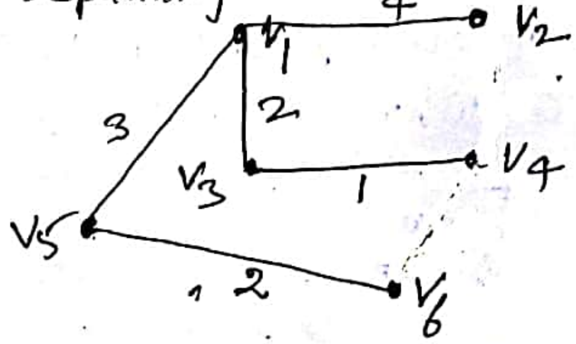
List all the edges with increasing order of the weight of edges

		weight
First edge —	(v_3, v_4)	1
Second edge —	(v_5, v_6)	2
Third edge —	(v_1, v_3)	2
Fourth edge —	(v_3, v_6)	3
fifth edge —	(v_1, v_5)	3
6th	(v_1, v_2)	4
7th ~~Sixth~~ edge	(v_2, v_4)	5
8th ~~Seventh~~ edge	(v_4, v_6)	6
9th edge	(v_3, v_5)	6

∴ Required minimal spanning tree T is given by



OR



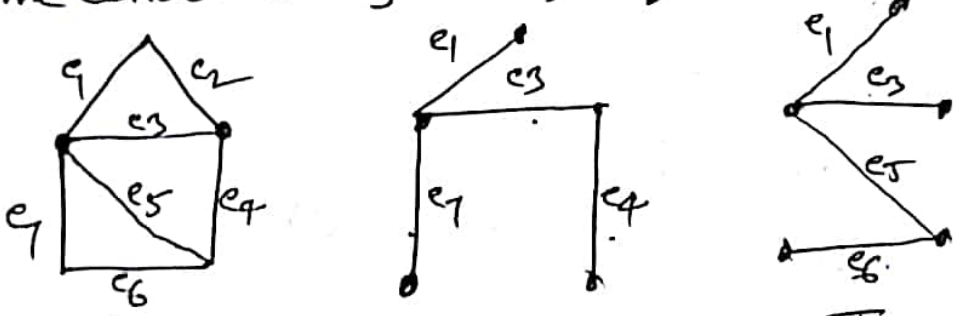
Total weight of T = 12 when is minimum

[इस method के अन्तर्गत हमें G की सभी vertices का प्रयोग करते हुए बिना circuit के connected graph बनाना होता है जिसको spanning tree T of G कहते है]

Distance between two spanning trees of a graph G

The distance between two spanning trees T_1 and T_2 of a graph G is defined as the number of edges of G present in one tree but not in the other. We denote it by $d(T_1, T_2)$.

Example



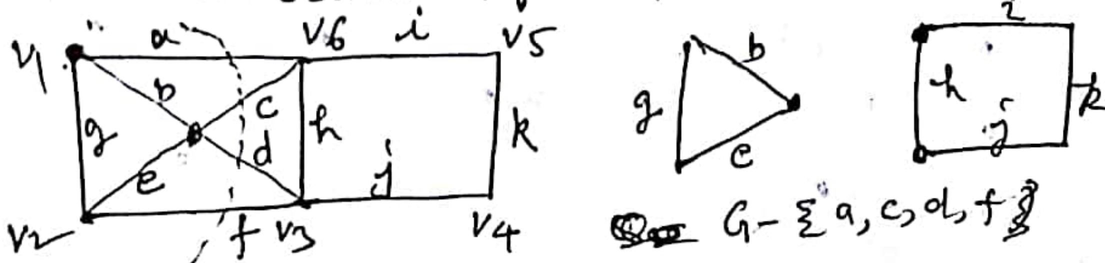
G T_1 T_2

$d(T_1, T_2) =$ no of edges of G present in T_1 but not in T_2

$= 02$ [e_7 and e_4 are not in T_2]

Cut-set. A cut set S of edges of a connected graph G is a minimal set of edges of G whose removal from G disconnect G.

Ex.



$G - \{a, c, d, f\}$

Here $S = \{a, c, d, f\}$

15. DDM

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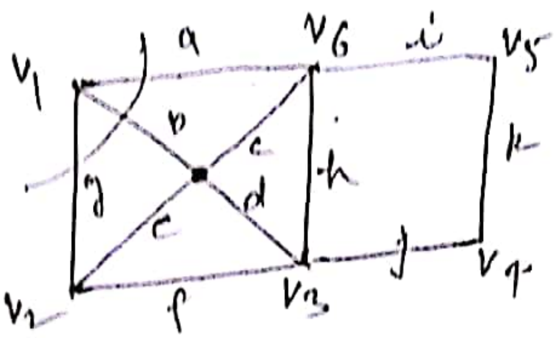
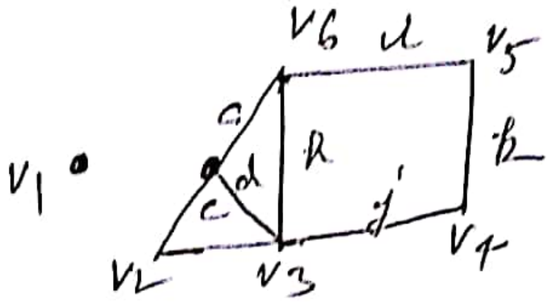
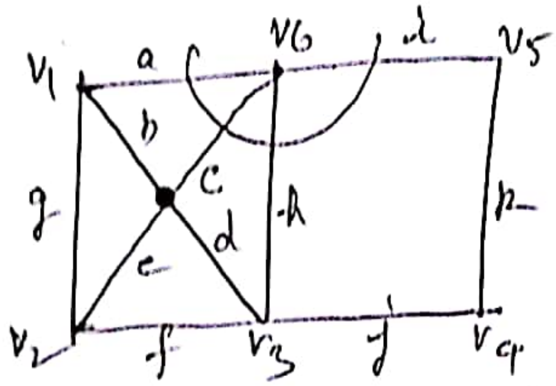
by Saban Kr Gupta



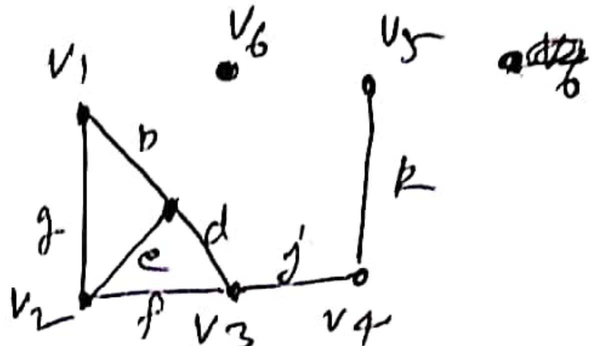
$S = (a, b, g)$



$G - \{a, b, g\}$



$S = \{a, c, h, i\}$



$G - \{a, c, h, i\}$